Cross ply and Angle ply laminates.

Laminate code :

A laminate is made of group of single layers bonded to each other. Each layer can be identified by

(i) Location in the laminate.

(2) Its Material

(3) Angle of orientation with referance axis.

Each lamina is represented by the angle of ply and seperated from other plies by a slash sign.

0
-45
90
60
30

[0/-45/90/60/30] — Laminate code

* 5 plies
* All are of different angle with referance axis.
* Same Material of same Thickness.

0
-45
90
90
60
0

$[0/-45/90_2/60/0]$

* 6 plies
* 2 90° plies adjacent to each other

$\Rightarrow 90_2 \Rightarrow$ No of adjacent plies of same angle.

0
-45
60
60

$[0/-45/60]_S \Rightarrow$ Symmetric laminate

* 6 plies
* plies above midsurface are of

and

0

-45

60

-45

0

$[0/-45/\overline{60}]_S$

* 5 plies.

$\xrightarrow{}$ Not repeated

60

↓

odd ply / 60° is mid ply orientation

Graphite/epoxy	0
Boron/epoxy	45
Boron/epoxy	-45
Boron/epoxy	-45
Boron/epoxy	45
Graphite/epoxy	0

$[0^{GR}/\pm 45^{B}]_S$

GR — Graphite

B — Boron

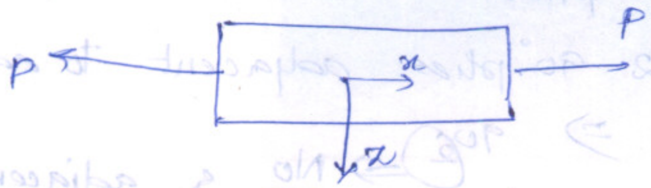
$\pm 45 \rightarrow$ first $+45^\circ$ and then -45° orientation

$S \rightarrow$ Symmetry.

[If $\mp 45 \rightarrow$ first -45° and then $+45^\circ$ orientation of laminates].

Stress - strain relations for a laminate:

1-D isotropic Beam



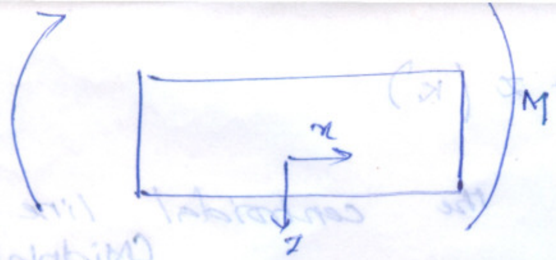
P — Axial load.

A — Cross sectional area.

$\sigma_{xx} = P/A$ = Normal stress

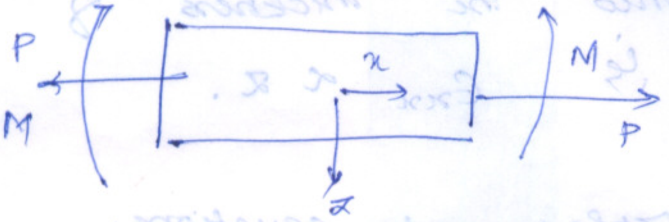
Normal Strain is

$\epsilon_{xx} = \dfrac{P}{AE}$



Pure Bending moment

The beam is assumed to be initially straight, after pure B.M, the beam undergoes twisting.



Axial load + B.M

To avoid twisting the Axial loads are applied which passes thro' the plane of symmetry, then a distance z from the centroidal line.

<u>Pure B.M</u> : $\epsilon_{xx} = \frac{z}{R}$ (Radius of curvature)

$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$

$\sigma/E = y/R$

i.e, $\epsilon = y/R$

$$\sigma_{xx} = \frac{Ez}{R}$$

$$= \frac{Mz}{I}$$

$$I = \int_A z^2 \, dA$$

$I = \int (\underbrace{y-\bar{y}}_{y=z})^2 \, dA$

<u>Axial load + B.M</u> :

$$\epsilon_{xx} = \frac{P}{AE} + z\left(\frac{1}{R}\right)$$

$$= \frac{P}{AE} + z \times \frac{M}{EI}$$

- Displacements are continuous and small throughout the laminate.

$|u|, |v|, |w| << $ Laminate Thickness (h)

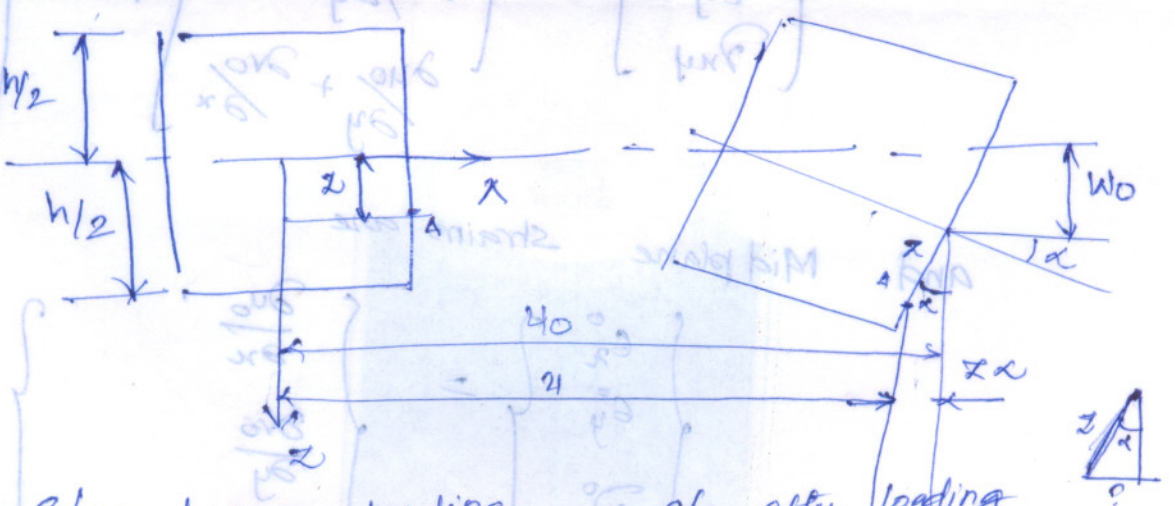
- ~~Each~~ lamina is elastic

- No slip occurs b/w the lamina interfaces

Consider a side view of a plate. The origin of plate is in mid plane i.e., $z = 0$.

Assume. u_o, v_o, w_o → displacements in x, y, z @ middle / midplane.

u, v, w — displacement @ any point



C/S before loading   C/S after loading

$\frac{x}{z} = \sin\alpha$

$x = \alpha z$

(α is small)

from fig.

$u = u_o - z\alpha$

and $\alpha = \frac{\partial w_o}{\partial x}$.

$$u = u_o - z\frac{\partial w_o}{\partial x}$$

If y-z plane, $V = V_0 - z \dfrac{\partial w_0}{\partial y}$

Normal strain
$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}$$
$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2}$$

Shear strain — $\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = \dfrac{\partial u_0}{\partial y} - z\dfrac{\partial^2 w_0}{\partial x \partial y} + \dfrac{\partial v_0}{\partial x} - z\dfrac{\partial^2 w_0}{\partial y \partial x}$

$$= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u_0/\partial x \\ \partial v_0/\partial y \\ \partial u_0/\partial y + \partial v_0/\partial x \end{Bmatrix} + z \begin{Bmatrix} -\partial^2 w_0/\partial x^2 \\ -\partial^2 w_0/\partial y^2 \\ -2\,\partial^2 w_0/\partial x\partial y \end{Bmatrix}$$

and Midplane strains are

$$\begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}$$

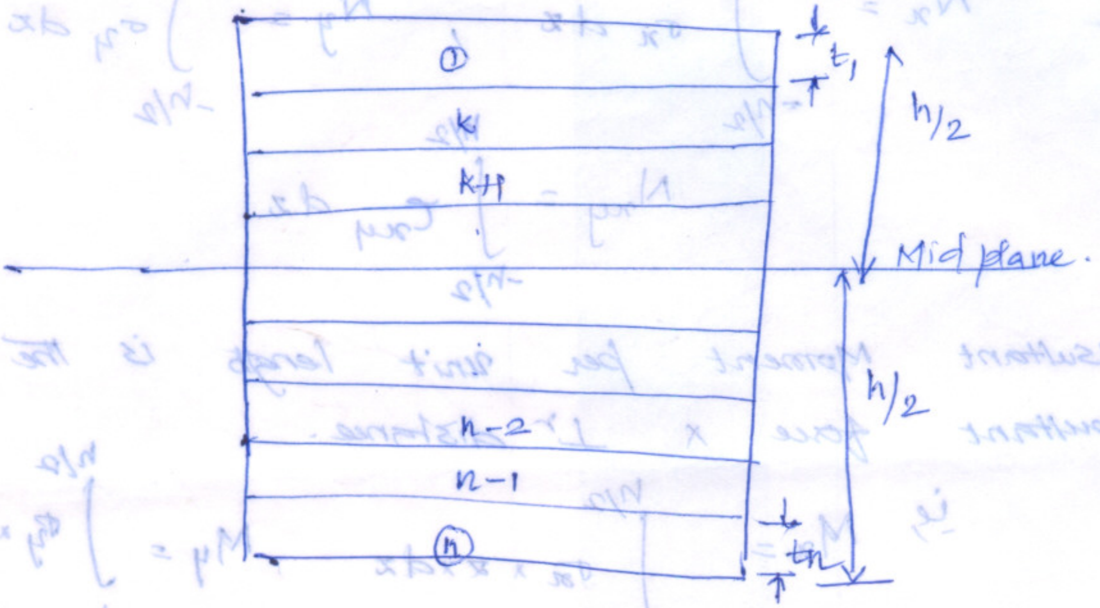
and Curvature

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} -\partial^2 w_0/\partial x^2 \\ -\partial^2 w_0/\partial y^2 \\ -2\,\partial^2 w_0/\partial x\partial y \end{Bmatrix}$$

$\therefore$ The laminate strains are,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} K_x \\ K_y \\ K_{xy} \end{Bmatrix}$$

Location of Mid plane :-



Ply ① :-

$$h_0 = \text{Top surface} = -h/2$$

$$h_1 = \text{Bottom surface} = -(h/2 - t_1)$$

$$= -h/2 + t_1$$ → ply.

−ve ⇒ Top to Bottom

ply (k) : $k = 2, 3, 4 \ldots\ldots\ n-3, n-2, n-1$.

$$h_{k-1} = -h/2 + \sum_{l=1}^{l=k-1} t_l \quad (\text{Top Surface})$$

$$h_k = -h/2 + \sum_{l=1}^{l=k} t_l \quad (\text{Bottom Surface})$$

ply (n) :-

$$h_{n-1} = h/2 - t_n \quad (\text{Top surface})$$

$$h_n = h/2 \quad (\text{Bottom Surface})$$

Stress & Strain in laminate

We know,

$$M_x = \int_z \sigma_x \times z \times dz \quad \text{and}$$

$$N_x = \int_z \sigma_x \times dz$$

Integrating the global stress in each lamina gives the resultant force per unit length in x-y plane thro' the laminate thickness 'h';

$$\therefore \quad N_x = \int_{-h/2}^{h/2} \sigma_x \, dz \; , \quad N_y = \int_{-h/2}^{h/2} \sigma_y \, dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz.$$

Resultant Moment per unit length is the resultant force $\times \perp^r$ distance.

i.e,

$$M_x = \int_{-h/2}^{h/2} \sigma_x \times z \times dz \quad M_y = \int_{-h/2}^{h/2} \sigma_y \times z \times dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, z \, dz.$$

N_x, N_y — Normal force / unit length

N_{xy} — Shear force / unit length

M_x, M_y — Bending moments / unit lengths

M_{xy} — Twisting moments / unit lengths.

$$\therefore \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z \, dz$$

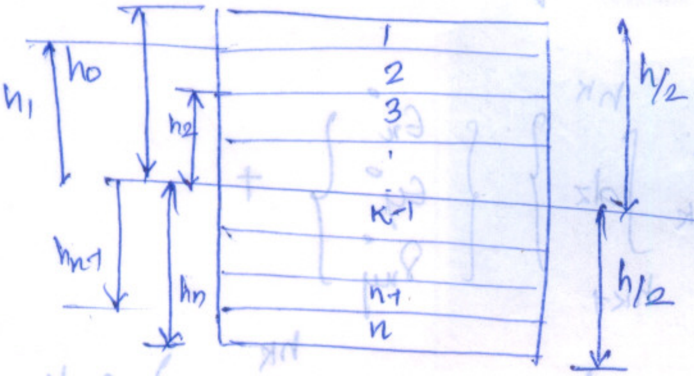
We know,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{16}} \\ \overline{Q_{12}} & \overline{Q_{22}} & \overline{Q_{26}} \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{66}} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

∴ By substituting the following relation,

We get,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\overline{Q}] \begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix} + z [\overline{Q}] \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$



The thickness of the laminate 'h' is

$$h = \sum_{k=1}^{n} t_k$$

force

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum^{n} \int^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z \, dz.$$

By subs. the stress/.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} [\bar{Q}] \begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix} dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} [\bar{Q}] \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} z \, dz$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} [\bar{Q}] \begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix} z \, dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} [\bar{Q}] \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} z^2 \, dz$$

$[\bar{Q}]$ is const. for each ply.

$$\therefore \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \left\{ \sum_{k=1}^{n} [\bar{Q}]_k \int_{h_{k-1}}^{h_k} dz \right\} \begin{Bmatrix} \epsilon_x^{\circ} \\ \epsilon_y^{\circ} \\ \gamma_{xy}^{\circ} \end{Bmatrix} + \left\{ \sum_{k=1}^{n} [\bar{Q}]_k \int_{h_{k-1}}^{h_k} z \, dz \right\} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy}\end{Bmatrix} = \left\{\sum_{k=1}^{n} [\bar{Q}]_k \int_{h_{k-1}}^{h_k} z\,dz\right\} \begin{Bmatrix}\epsilon_x^\circ\\ \epsilon_y^\circ\\ \gamma_{xy}^\circ\end{Bmatrix} +$$

$$\left\{\sum_{k=1}^{n} [\bar{Q}]_k \int_{h_{k-1}}^{h_k} z^2\,dz\right\}\begin{Bmatrix}k_x\\ k_y\\ k_{xy}\end{Bmatrix}$$

~~Take~~, We know,

$$\int_{h_{k-1}}^{h_k} dz = h_k - h_{k-1}$$

$$\int_{h_{k-1}}^{h_k} z\,dz = \frac{1}{2}\left[h_k^2 - h_{k-1}^2\right]$$

$$\int_{h_{k-1}}^{h_k} z^2\,dz = \frac{1}{3}\left[h_k^2 - h_{k-1}^2\right]$$

Take,

Shear extension coupling.

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy}\end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \boxed{A_{16}} \\ A_{12} & A_{22} & \boxed{A_{26}} \\ \boxed{A_{16}} & \boxed{A_{26}} & A_{66}\end{bmatrix} \begin{Bmatrix}\epsilon_x^\circ\\ \epsilon_y^\circ\\ \gamma_{xy}^\circ\end{Bmatrix} +$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} +$$

$$\begin{bmatrix} D_{11} & D_{12} & \boxed{D_{16}} \\ D_{12} & D_{22} & \boxed{D_{26}} \\ \boxed{D_{16} \quad D_{26}} & & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$

Bending twist coupling

where,

$$A_{ij} = \sum_{k=1}^{n} \left[(\bar{Q}_{ij}) \right]_k (h_k - h_{k-1})$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [\bar{Q}_{ij}]_k (h_k^2 - h_{k-1}^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [\bar{Q}_{ij}]_k (h_k^3 - h_{k-1}^3)$$

$i = 1, 2, 3 \ldots \quad j = 1, 2 \ldots$

[A] - Extensional Stiffness Matrix

[B] - Coupling Stiffness matrix

[D] - Bending Stiffness Matrix.

[A] relates resultant inplane forces to inplane strains.

[D] relates resultant bending moment to mid plane curvatures.